Newtonian N-body simulations are compatible with cosmological perturbation theory

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Contrary to recent claims [1], Newtonian N-body simulations of collisionless Dark Matter in a ΛCDM background are compatible with general relativity and are not invalidated by general relativistic effects at the linear level. This verdict is based on four facts. (1) The system of linearized Einstein equations and conservation laws is well-posed in the gauge invariant formulation and physically meaningful. (2) Comparing general relativity with its Newtonian approximation at a given order in perturbation theory is only meaningful at the level of observables. (3) The dynamics of observables describing a dust fluid in general relativity and its Newtonian approximation agree at the linear level. Any disagreement for observables on the lightcone are well-known, of which the most dominant is gravitational lensing. (4) Large fluctuations in the Hubble parameter contribute significantly only to gravitational lensing effects. Therefore, these fluctuations are not in conflict with Newtonian N-body simulations beyond what has already been carefully taken into account using ray tracing technology.

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I. INTRODUCTION

There seems to be little doubt that the gravitational formation of structures on scales deep inside the Hubble volume is accurately described by Newtonian gravity, with general-relativistic effects entering at subleading level in the perturbative description. Nevertheless it was claimed [1] recently that effects in cosmological perturbation theory become dominant over 2nd order effects in the Newtonian approximation on scales larger than 10 Mpc.

In greater detail the argument was based on the following. The evolution of cosmological perturbations was calculated in a particular coordinate system, called the *Newtonian matter gauge* (NM), in which the linear density contrast $\delta_{\rm NM}$ and the peculiar velocity $v_{\rm NM}$ coincide with the corresponding quantities $\delta_{\rm N} = \delta_{\rm NM}$, $v_{\rm N} = v_{\rm NM}$ in the Newtonian (N) approximation. It was found that fluctuations in the local Hubble parameter $\delta^H_{\rm NM} \equiv -1 - K_{\rm NM}/(3H)$, where $K_{\rm NM}$ denotes the extrinsic curvature, are of order $\mathcal{O}(\delta_{\rm NM})$. The significance of this observation was evaluated by comparing $\delta^H_{\rm NM}$ to the size of second order corrections $\delta^{(2)}_{\rm N}$ in the Newtonian approximation. For comoving scales k and redshifts z with

$$\delta^H_{\rm NM} \geq \delta^{(2)}_{\rm N} \qquad \text{and} \qquad \delta^{(2)}_{\rm N} \ll \delta_{\rm N} < 1\,, \tag{1}$$

it was argued that relativistic effects linear in cosmological fluctuations dominate over nonlinear Newtonian effects well within the domain of validity of perturbation theory. Based on this criterion, the authors of [1] found that linear cosmological perturbations on an Einstein-de Sitter background dominate over Newtonian nonlinearities for scales $k^{-1} > 10$ Mpc during the redshift interval $z \in [0.4, 750]$, from which they concluded that Newtonian N-body simulations cannot be trusted on these scales during the specified redshift interval.

Since by choice of gauge, $\delta_{\rm N} = \delta_{\rm NM}$, $v_{\rm N} = v_{\rm NM}$, while the fluctuations in the Hubble parameter are absent in the Newtonian approximation, and thus in N-body simulations, it is interesting to ask how these fluctuations become manifest in observables? It might be expected [1] that fluctuations in the Hubble parameter cause additional redshift space distortions, because the observed redshift depends on $K_{\rm NM}$ integrated along the line of sight between observer and source, in addition to the peculiar velocities of observer and source, $v^{\rm O}_{\rm NM}$ and $v^{\rm S}_{\rm NM}$, respectively. We find indeed that these fluctuations contribute considerably to redshift space distortions, however, only via gravitational lensing, which is well known as *lensing magnification*, e.g [2–8]. These lensing induced distortions are taken into account in N-body simulations using ray tracing technology [9, 10].

Cosmological perturbation theory can be compared to its Newtonian approximation in a meaningful way only by comparing observables. Observables are by definition gauge invariant combinations of the perturbations. The set of observables should be specified before a choice of gauge is implemented, after the gauge redundancies have been removed it is impossible to identify observables. However, once the observables have been identified, gauge freedom is not sacrosanct and can either be removed or, equivalently, used to rewrite the theory using gauge invariant variables. Both procedures are perfectly valid. A comparison of observables using gauge invariant perturbations has been performed in [11] and was criticized in [1] as follows: "The initial conditions must be specified on a spatial Cauchy hypersurface, which in the context of cosmological perturbation theory corresponds to a particular foliation of space-time, i.e., to a hypothetical observer

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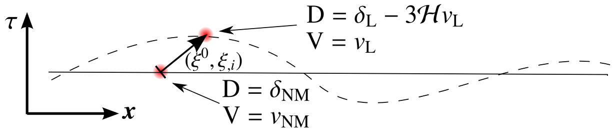

FIG. 1. D and V agree on different hypersurfaces (*full* Newton matter gauge and *dashed* longitudinal gauge) evaluated at the same coordinate values in their respective coordinate systems. Only the physical interpretation of D and V changes and on both hypersurfaces an hypothetical observer could determine D and V.

who is able to determine physical quantities on a spatial hypersurface. The relativistic-Newtonian correspondence mixes the quantities defined on different spatial hypersurfaces and thus no hypothetical observer in the Einsteinian world could actually determine these combined quantities."

This is a misconception that requires immediate clarification. Although gauge invariant variables might have a simple physical interpretation only in one specific gauge, they present observables in all other coordinate systems and, in particular, different hypersurfaces, as well. For instance, let D be a gauge invariant variable which reduces to the linear density contrast δ_{S} in synchronous (S) gauge, measured by an observer at rest with respect to synchronous coordinates. Let V denote a gauge invariant variable which reduces to the peculiar velocity v_{L} as measured by an observer at rest relative to the longitudinal coordinate system. This is the situation referred to "defined on different spatial hypersurfaces" in the above quote. However, (D,V) are defined in all possible coordinate systems and observers adopted to different and arbitrary coordinate systems can measure D and V and will find the same numerical values for them. By construction, it is not the definition of gauge invariant variables that is tied to certain hypersurfaces, but it is their physical interpretation. This is why any smart observer will construct the set of observables before choosing a particular gauge to measure them, because even if not all observables have a convenient physical interpretation in the observer's coordinate frame, they still resemble the only meaningful quantities for any other observer. The only reasonable academic debate between observers adopted to different coordinate systems is about the physical interpretation of the gauge invariant variables.

As an example, consider an observer who is adopted to a longitudinal (L) coordinates system. This observer will interpret D as $\mathrm{D}_{\mathrm{L}} = \delta_{\mathrm{L}} - 3\mathcal{H}v_{\mathrm{L}}$. Although the observer has a physical interpretation for the gauge dependent quantities $(\delta_{\mathrm{L}}, v_{\mathrm{L}})$, (s)he understands the necessity to construct gauge invariant combinations involving $(\delta_{\mathrm{L}}, v_{\mathrm{L}})$, rather than assuming any other observer adopted to an arbitrary coordinate system would agree on the values of the gauge dependent quantities. The observer adopted to the longitudinal frame can measure $(\delta_{\mathrm{L}}, v_{\mathrm{L}})$ on a hypersurface, set up the initial value problem for an appropriate evolution equation involving (D,V) and finally solve for them, instead of $(\delta_{\mathrm{L}}, v_{\mathrm{L}})$, see Fig. 1.

The plan for the rest of this paper is as follows. In Section II it is shown explicitly that the hypothetical observables $(\delta_{\mathrm{N}}, v_{\mathrm{N}})$ of linearized Newtonian gravity and (D,V) of linearized Einsteinian gravity obey the same evolution equations on a flat ΛCDM background (in [1] only $\Lambda = 0$ has been considered). This was expected since (D,V) reduce to $(\delta_{\mathrm{NM}}, v_{\mathrm{NM}})$ for observers at rest in the Newtonian matter coordinate system.

However, a real observer cannot measure (D,V) on a hypersurface, because (s)he can only observe the light cone associated with (her) him via light rays, which get affected by curvature perturbations. This induces a feedback of general relativistic effects on (D,V), although (D,V) still coincide with $(\delta_{\mathrm{NM}}, v_{\mathrm{NM}})$ on each hypersurface. Note that the status of (D,V) as observables is not challenged by the practical obstacles that prevent any observer from measuring them on the entire hypersurface.

Section III is devoted to investigate the impact of Hubble parameter fluctuations on the linear density contrast observed along a light cone. This is a well-known example highlighting the fact that δ^{H}_{NM} contributes significantly only to gravitational lensing. The strongest additional redshift space distortions due to gravitational lensing [2–8] originate indeed from δ^{H}_{NM} (as well as from intrinsic curvature, see Section III). The gauge invariant lensing term, of course, does not depend on the gauge the observer preferred. An observer adopted to the longitudinal gauge finds a negligible contribution from δ^{H}_{L} to gravitational lensing.

Our main result is that Newtonian N-body simulations are in congruence with cosmological perturbation theory and are not threatened by relativistic effects at the linear level although relativistic effects can become significant. We give a pratical dictionary to use N-body simulation data to evaluate these corrections.

II. CONGRUENCE OF LINEAR OBSERVABLES ON A HYPERSURFACE

In this section we show that a pressureless fluid in a universe with ΛCDM background geometry can be characterized by observables (D,V) in 1st-order cosmological perturbation theory that obeys evolution equations identical to those governing the evolution of $(\delta_{\mathrm{N}}, v_{\mathrm{N}})$.

Restricting attention solely to scalar perturbations, the conformal evolution equations for $(\phi_{\mathrm{N}}, \delta_{\mathrm{N}}, v_{\mathrm{N}})$ in the Newtonian approximation are given by

$$\Delta\phi_{\mathrm{N}} = \tfrac{3}{2}\mathcal{H}^2\Omega_{\mathrm{m}}\delta_{\mathrm{N}} \,, \tag{2a}$$
$$\delta'_{\mathrm{N}} + \Delta v_{\mathrm{N}} = 0 \,, \tag{2b}$$
$$v'_{\mathrm{N}} + \mathcal{H}v_{\mathrm{N}} = -\phi_{\mathrm{N}} \,, \tag{2c}$$

with $\mathcal{H}$ denoting the conformal expansion rate of the background determined by Friedmann equations

$$3\mathcal{H}^2 = 8\pi G(\bar{\varrho}_{\mathrm{m}} + \varrho_{\Lambda})a^4 \,, \tag{3a}$$
$$4\pi G a^2\bar{\varrho}_{\mathrm{m}} = \tfrac{3}{2}\Omega_{\mathrm{m}}\mathcal{H}^2 = a^2(\mathcal{H}^2 - \mathcal{H}') \,, \tag{3b}$$

where $\bar{\varrho}_{\mathrm{m}} \propto a^{-3}$ and Ω_{m} denotes the background matter density relative to the critical density.

The Newtonian perturbation variables $(\delta_{\mathrm{N}}, v_{\mathrm{N}})$ are defined by the dark matter density $\varrho_{\mathrm{m}} = \bar{\varrho}_{\mathrm{m}}(1 + \delta_{\mathrm{N}})$ and the peculiar

velocity $\boldsymbol{v}_\mathrm{N} = \nabla v_\mathrm{N}$. The triple $(\phi_\mathrm{N}, \delta_\mathrm{N}, v_\mathrm{N})$ constitutes the set of observables relevant for the discussion.

The corresponding description in general relativity requires a background metric around which the geometry fluctuates. Restricting attention again solely to scalar metric fluctuations, (ϕ, w, ψ, h), the total metric field is give by

$$\mathrm{d}s^2 = a^2\Big[-(1+2\phi)\,\mathrm{d}\tau\otimes\mathrm{d}\tau + 2w_{,i}\,\mathrm{d}\tau\otimes\mathrm{d}x^i + \\ + [(1-2\psi)\delta_{ij} + 2h_{,ij}]\mathrm{d}x^i\otimes\mathrm{d}x^j\Big]\,. \tag{4}$$

The total pressureless source is encoded in $T = \varrho_\mathrm{m} U\otimes U$, with $\varrho_\mathrm{m} = \bar\varrho_\mathrm{m}(1+\delta)$ and $U = (1-\phi, \nabla v)/a$. Altogether the dynamical degrees of freedom (ϕ,ψ,w,h,δ,v) are the gauge dependent metric, density and velocity perturbations.

Observables can be constructed by the following gauge invariant linear combinations:

$$\Phi = \phi + [(w-h')a]'/a\,, \tag{5a}$$
$$\Psi = \psi - \mathcal{H}(w-h')\,, \tag{5b}$$
$$\mathrm{D} = \delta - 3\mathcal{H}(v+w)\,, \tag{5c}$$
$$\mathrm{V} = v + h'\,. \tag{5d}$$

The perturbed Einstein and conservation equations then yield evolution equations [11] for the gauge invariant quantities $(\Phi,\Psi,\mathrm{D},\mathrm{V})$:

$$\Delta\Phi = \tfrac{3}{2}\mathcal{H}^2\Omega_\mathrm{m}\mathrm{D}\,, \tag{6a}$$
$$\mathrm{D}' + \Delta\mathrm{V} = 0\,, \tag{6b}$$
$$\mathrm{V}' + \mathcal{H}\mathrm{V} = -\Phi\,, \tag{6c}$$

where the background equations (3) and the linearized $(0j)$- and (ij)-Einstein equations have been used:

$$\mathcal{H}\Phi + \Psi' = -\tfrac{3}{2}\mathcal{H}^2\Omega_\mathrm{m}\mathrm{V}\,, \tag{7a}$$
$$\Phi = \Psi\,. \tag{7b}$$

Comparing (6) with the Newtonian approximation (2) it is evident that the evolution equations are identical in form. In additions, $(\Phi,\mathrm{D},\mathrm{V})$ constitute the triple of observables relevant for this discussion. Of course, any linear combination of these gauge invariant variables constitutes an equally legitimate observable. The triple $(\Phi,\mathrm{D},\mathrm{V})$ is favored only to establish directly the correspondence between relativistic observables and observables in the Newtonian approximation at the linear level.

Note that (Φ,Ψ) has the same quasi-static dynamics as ϕ_N, which allow us to qualify relativistic corrections to Newtonian observables, e.g. (8) below, as large or small in comparison to 2nd order corrections in the Newtonian approximation (1).

Let us emphasize again that there is no gauge $\mathcal{G}$ in which simultaneously

$$\Phi = \phi_\mathcal{G},\ \ \Psi = \psi_\mathcal{G},\ \ \mathrm{D} = \delta_\mathcal{G},\ \ \mathrm{V} = v_\mathcal{G}\,.$$

Different observers simply assign different physical meaning to these gauge invariant variables. For instance, in longitudinal, synchronous and Newtonian matter gauge the following physical interpretations hold:

Gauge	Φ	Ψ	D	V
L	ϕ_L	ψ_L	$\delta_\mathrm{L} - 3\mathcal{H}v_\mathrm{L}$	v_L
S	$-h''_\mathrm{S} - h'_\mathrm{S}\mathcal{H}$	$\psi_\mathrm{S} + h'_\mathrm{S}\mathcal{H}$	δ_S	h'_S
NM	$-v'_\mathrm{NM} - v_\mathrm{NM}\mathcal{H}$	$\psi_\mathrm{NM} + v_\mathrm{NM}\mathcal{H}$	δ_NM	v_NM

III. LINEAR OBSERVABLES ON THE LIGHTCONE

A physical observer is in practice not able to measure (D,V) everywhere on any hypersurface. Instead, a physical observer can only learn about (D,V) by employing light rays traveling along (her) his respective light cone. As a consequence of such an observation campaign, (D,V) become subject to relativistic effects that have no Newtonian counterpart. The light rays will be gravitationally lensed and these lensing effects will be attributed to (D,V). Since gravitational lenses are absent in the Newtonian approximation, the dictionary $(\Phi,\mathrm{D},\mathrm{V}) \leftrightarrow (\phi_\mathrm{N},\delta_\mathrm{N},v_\mathrm{N})$ is challenged.

As an example, consider the linear density fluctuation $\Delta(\boldsymbol{n},z)$ at the observed redshift z and direction $-\boldsymbol{n}$ on the sky or, equivalently, in the direction $\boldsymbol{n}$ of the incoming light ray at the observer's space-time position $(\tau_\mathrm{O},\boldsymbol{x}_\mathrm{O})$, within the relativistic framework of 1st order cosmological perturbation theory. It is given by the gauge invariant expression [7]

$$\Delta(\boldsymbol{n},z) = \mathrm{D} - \frac{1}{\mathcal{H}}\partial_r^2\mathrm{V} - \frac{1}{r_S}\int_0^{r_S} d\lambda\frac{r_S - r}{r}\Delta_\Omega(\Phi+\Psi) + \\ + \left(\frac{\mathcal{H}'}{\mathcal{H}^2} + \frac{2}{r_S\mathcal{H}}\right)\left(\Psi - \partial_r\mathrm{V} + \int_0^{r_S} d\lambda(\Phi+\Psi)'\right) \\ + \frac{1}{\mathcal{H}}\Phi' + 3\mathcal{H}\mathrm{V} - 2\Phi + \Psi + \frac{2}{r_S}\int_0^{r_S} d\lambda(\Phi+\Psi)\,, \tag{8}$$

where all functions are evaluated along the unperturbed light cone $\boldsymbol{x} = \boldsymbol{x}_\mathrm{O} - \boldsymbol{n}r(z)$, $\tau = \tau_\mathrm{O} - r(z)$, with $r(z) = \int_0^z dz'/(H(z')a_\mathrm{O})$, the unperturbed affine parameter $\lambda = r$ and S denotes the source. Δ_Ω is the angular part of the Laplacian in spherical coordinates. Detailed derivations of (8) can be for instance found in [4, 7, 8].

The Newtonian approximation of gravity is void of the light cone concept. Moreover, the gravitational potential couples only to massive bodies, in particular, there is no coupling to light rays. In other words, light rays cannot probe ϕ_N. The assumption that there is a light cone embedded in the background cosmology, albeit an artificial point of view, allows to obtain the first and second term of (8) in the Newtonian approximation, $\Delta_\mathrm{N}(\boldsymbol{n},z) = \delta_\mathrm{N} - \mathcal{H}^{-1}\partial_r^2 v_\mathrm{N}$. The second term is known as *Kaiser-effect* [12] and is the dominant redshift space distortion for small redshifts. Moreover, let us assume that light couples to ϕ_N such that its bending around the Sun conforms to actual observations. Including the lensing contribution,

$$\Delta_\mathrm{N}(\boldsymbol{n},z) = \delta_\mathrm{N} - \mathcal{H}^{-2}\partial_r^2 v_\mathrm{N} - \frac{1}{r_S}\int_0^{r_S} d\lambda\frac{r_S - r}{r}\Delta_\Omega\phi_\mathrm{N}\,. \tag{9}$$

Depending on scale, redshift and redshift binning, the lensing contribution can be the leading redshift space distortion,

which can even dominate over $\delta_{\rm N}$ [7, 8] for sufficiently distant sources.

For a more transparent treatment, let us define a Newtonian observable $\Delta_{\rm N}(\boldsymbol{n}, z)$ through the following replacements in the relativistic quantity (8):

$$\Delta_{\rm N}(\boldsymbol{n}, z) \equiv \Delta(\boldsymbol{n}, z)\Big|_{{\rm D}\to\delta_{\rm N}, {\rm V}\to v_{\rm N}, \Phi=\Psi\to\phi_{\rm N}} . \tag{10}$$

Using the results from the last section it follows that $\Delta_{\rm N}(\boldsymbol{n}, z) = \Delta(\boldsymbol{n}, z)$. As a consequence, *N-body simulations can be used directly to extract relativistic observables* when scalar dust fluctuations on a ΛCDM background are considered at the linear level.

Let us comment on why fluctuations $\delta^H_{\rm NM}$ in the Hubble parameter $\bar{K} = -3H$,

$$\begin{aligned}\delta^H &\equiv -\frac{\delta K/3}{\mathcal{H}/a} \\ &= \frac{-1}{\mathcal{H}}\left(\psi' + \mathcal{H}\phi + 1/3\,\Delta(w - h')\right) ,\end{aligned} \tag{11}$$

are, in fact, strongly contributing only to the lensing term and, therefore, were identified correctly in [1] as a major source of redshift space distortions. Clearly, large fluctuations in the Hubble parameter do not imply that Newtonian N-body simulations cannot be trusted. Instead it implies that either (9) or (10) (or, better, an expression including nonlinear effects) should be used to compare numerical experiments based on the Newtonian approximation to observations, which was well known [2–8] before the work [1].

We could have argued based on gauge invariance alone [3, 7] that $\Delta(\boldsymbol{n}, z)$ is constructed from extrinsic curvature δ^H, intrinsic curvature $R^{(3)} = 4\Delta\psi/a^2$, anisotropic extrinsic curvature $A_{ij} = a(\partial_i\partial_j/\Delta - \delta_{ij}/3)\Delta(w - h')$, and the divergence of the observer's coordinate acceleration $\boldsymbol{a} = \nabla \ln[a(1 + \phi)]$ in such a way that arguing about the size of δ^H adapted to various gauges is meaningless. In certain gauges δ^H might qualify as large (NM and S), while in others it qualifies as small (L), but this does not matter at all.

Any change in δ^H induced by a transition between coordinate systems is compensated for by corresponding changes in $\delta^R \equiv \Delta\psi$, $\delta^{\boldsymbol{a}} \equiv \Delta\phi$ and $\delta^A \equiv \Delta(w - h')$. This can be checked explicitly for the gauge invariant lensing term $\Delta(\Phi+\Psi)$, which can be deconstructed into the various gauge dependent curvatures as follows:

$$\Delta(\Phi + \Psi) = \delta^R + \delta^{\boldsymbol{a}} + \delta^{A'} . \tag{12}$$

Note that δ^H contains the anisotropic extrinsic curvature δ^A. In the Newtonian matter gauge, $\delta^H_{\rm NM}$ becomes large just because $\delta^A_{\rm NM}$ is large, while in the longitudinal gauge $\delta^A_{\rm L} = 0$ and $\phi_{\rm L}$, $\psi_{\rm L}$ are quasi-static, meaning they basically remain at their initially small values (except close to neutron stars and black holes, see [13, 14] for more details). As a consequence, $\delta^H_{\rm L}$ is negligible. Since $\phi_{\rm L} = \Phi$ and $\psi_{\rm L} = \Psi$, all other terms in (8) involving Ψ and Φ, as well as their conformal time derivatives remain much smaller than the first three terms in (8). It can be shown that whenever $\delta^H_{\rm NM}$ contributes to these less relevant terms, its $\delta^A_{\rm NM}$ component is either compensated for by another $\delta^A_{\rm NM}$ contribution or rendered harmless by Δ^{-1}.

The discussion outlined for $\Delta(\boldsymbol{n}, z)$ applies quite generally to any observable. A constructive algorithm for an arbitrary observable is the following: (*i*) construct the general relativistic gauge invariant observable and express it in terms of (D,V) and Φ. (*ii*) Use the quasi-static evolution of Φ to determine which contributions qualify as large. If one can identify large contributions that are not reflected in the corresponding Newtonian observable, then these contributions will have a genuine relativistic origin. (*iii*) Employ the dictionary $({\rm D}, {\rm V}, \Phi) \to (\delta_{\rm N}, v_{\rm N}, \phi_{\rm N})$ to extract relativistic observables using Newtonian N-body simulations. This has been worked out in much greater in detail in [15, 16].

IV. CONCLUSION

In summary, we have shown that fluctuations in the Hubble parameter do not give rise to new, so far overlooked redshift space distortions. They contribute to redshift space distortions, however only to the well-known lensing magnification. Therefore contrary to the claims of [1], Newtonian N-body simulations *are* the appropriate numerical experiments to extract information on relativistic observables at least in 1st order perturbation theory.

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